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The method of boundary integral equations for a mixed scattering problem

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ABSTRACT

In this paper we consider the scattering of an electromagnetic time-harmonic plane wave by an infinite cylinder having an open arc and a bounded domain in R^2 as cross section. To this end, we solve a scattering problem for the Helmholtz equation in R^2 where the scattering object is a combination of a crack Γ and a bounded obstacle D, and we have Dirichlet-impedance type boundary condition on Γ and Dirichlet boundary condition on ∂D ($\partial D \in C^2$). Applying potential theory, the problem can be reformulated as a boundary integral system. We establish the existence and uniqueness of a solution to the system by using the Fredholm theory.

1. Introduction

The direct and inverse scattering problems for cracks were initiated by Kress in 1995 [14]. In the paper, Kress considered the direct and inverse scattering problem for a perfectly conducting crack, and used integral equation method to solve both the direct and inverse problems for a sound-soft crack. The scattering problem in the unbounded domain is thus converted into a boundary integral equation. Mönch [16] extended this approach to a Neumann crack in 1997. In 2000, Kress's work was continued by Kirsch and Ritter in [12] who used the factorization method to reconstruct the shape of the crack from the knowledge of the far-field pattern, and in the same year these results were generalized to the scattering problem with cracks for Maxwell equations in [1] by Ammari, Bao and Wood. Later in 2003, F. Cakoni and D.L. Colton in [3] discussed the direct and inverse scattering problems for cracks (possibly) coated on one side by a material with surface impedance λ. Extending to the

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impedance problem, Kuo-Ming Lee [15] considered the direct and inverse scattering problem for an impedance crack in 2008. However, studying an inverse problem always requires a solid knowledge of the corresponding direct problem. Therefore, in this paper we just consider the direct scattering problem for a combination of a crack Γ and a bounded obstacle D ($\partial D \in C^2$), and the corresponding inverse scattering problem can be considered by similar methods in [6–12,18,19] and the reference therein.

Briefly speaking, in this paper we consider the scattering of an electromagnetic time-harmonic plane wave by an infinite cylinder having an open arc Γ and a bounded domain D ($\partial D \in C^2$) in R^2 as cross section. We assume that the crack is coated on one side by a material with surface impedance λ. This corresponds to the situation when the boundary or more generally a portion of the boundary is coated with an unknown material in order to avoid detection. Assuming that the electric field is polarized in the TM mode, this leads to a mixed boundary value problem for the Helmholtz equation defined in the exterior of an open arc and a bounded domain in R^2.

The goal of this paper is to establish the existence and uniqueness of a solution to this direct scattering problem. As is known, the method of boundary integral equations is suitable for various direct scattering problem with nice boundary (see [4,5,13] and the reference therein). A few authors have applied such method to study the scattering problem with Lipschitz boundary and mixed boundary conditions. In this paper, we will use the method of boundary integral equations to consider the problem (3) (see Section 2), and obtain the existence and uniqueness of the solution by using the Fredholm theory. The difficult thing is to show the boundary integral operator A is a Fredholm operator with index zero since the boundary is a combination of an arc and a closed curve and we have a complicated mixed boundary conditions.

This paper is organized as follows. In Section 2, we will introduce the direct scattering problem, establish uniqueness to the problem and reformulate the problem as a boundary integral system by using single- and double-layer potentials. In Section 3, we will show the main results on the existence and uniqueness of the solution to the boundary integral system by using the potential theory and Fredholm theory.

2. Formulation of the scattering problem

Consider the scattering of time-harmonic electromagnetic plane waves from an infinite cylinder having an open arc Γ (smooth) and a bounded domain D ($\partial D \in C^2$) in R^2 as cross section. For further considerations we extend the arc Γ to an arbitrary smooth, simply connected, closed curve $\partial\Omega$ ($\partial\Omega \in C^2$) enclosing a bounded domain Ω such that the normal vector ν on Γ coincides with the normal vector on $\partial\Omega$ which we denote by ν. Notice that ν is the unit normal vector defined almost everywhere on $\partial\Omega$ and ∂D (except a finite number of points) and direct into the exterior of Ω and D. We assume that the domain D is completely contained in Ω, i.e., $D \subset \Omega$ and $\partial D \cap \partial\Omega = \emptyset$. Throughout this paper, we shall assume that $\partial D \in C^2$ and $\partial\Omega \in C^2$.

The arc Γ can be parameterized as

$$\Gamma = \left\{z(s):\ s \in [s_0, s_1]\right\}$$

where $z: [s_0, s_1] \to R^2$ is an injective piecewise C^1 function. We denote the outside of Γ with respect to the chosen orientation by Γ^+ and inside by Γ^-.

Dirichlet-impedance type boundary condition on Γ and Dirichlet boundary condition on ∂D lead to the following problem:

$$\begin{cases} \Delta U + k^2 U = 0 & \text{in } R^2 \backslash (\bar{D} \cup \Gamma), \\ U_- = 0 & \text{on } \Gamma^-, \\ \dfrac{\partial U_+}{\partial \nu} + ik\lambda U_+ = 0 & \text{on } \Gamma^+, \\ U = 0 & \text{on } \partial D, \end{cases} \tag{1}$$

where the wavenumber k and impedance coefficients are positive, i.e. $k>0$ and $\lambda>0$, and $-k^2$ is not Dirichlet eigenvalue of the Laplace operator in D. Notice that $U_\pm(x)=\lim_{h\to 0^+}U(x\pm h\nu)$ for $x\in\Gamma$ and $\frac{\partial U_\pm}{\partial\nu}=\lim_{h\to 0^+}\nu\cdot\nabla U(x\pm h\nu)$ for $x\in\Gamma$. The total field $U=u^i+u$ is decomposed into the given incident field $u^i(x)=e^{ikx\cdot d}$, $|d|=1$, and the unknown scattered field u which is required to satisfy the Sommerfeld radiation condition

$$\lim_{\nu\to\infty}\sqrt{r}\left(\frac{\partial u}{\partial r}-iku\right)=0 \tag{2}$$

uniformly in $\hat{x}=x/|x|$ with $r=|x|$.

Consider the scattered field u, then the problem (1) and (2) is a special case of the following problem:

Given $g\in H^{1/2}(\Gamma^-)$, $h\in H^{-1/2}(\Gamma^+)$ and $f\in H^{1/2}(\partial D)$, find $u\in H^1_{loc}(R^2\backslash(\bar{D}\cup\bar{\Gamma}))$ such that

$$\begin{cases}\Delta u+k^2u=0 & \text{in } R^2\backslash(\bar{D}\cup\Gamma),\\ u_-=g & \text{on } \Gamma^-,\\ \dfrac{\partial u_+}{\partial\nu}+ik\lambda u_+=h & \text{on } \Gamma^+,\\ u=f & \text{on } \partial D,\end{cases} \tag{3}$$

and u is required to satisfy the Sommerfeld radiation condition (2).

Here $H^1_{loc}(R^2\backslash(\bar{D}\cup\bar{\Gamma}))$, $H^{1/2}(\partial D)$, $H^{1/2}(\Gamma^-)$ and $H^{-1/2}(\Gamma^+)$ are the usual Sobolev spaces. We define the following spaces (see [3]):

$$H^{1/2}(\Gamma)=\{u|_\Gamma:\ u\in H^{1/2}(\partial\Omega)\}, \tag{4}$$

$$\tilde{H}^{1/2}(\Gamma)=\{u\in H^{1/2}(\Gamma):\ \operatorname{supp}u\subseteq\bar{\Gamma}\}, \tag{5}$$

$$H^{-1/2}(\Gamma)=\left(\tilde{H}^{1/2}(\Gamma)\right)',\quad \text{the dual space of } \tilde{H}^{1/2}(\Gamma), \tag{6}$$

$$\tilde{H}^{-1/2}(\Gamma)=\left(H^{1/2}(\Gamma)\right)',\quad \text{the dual space of } H^{1/2}(\Gamma) \tag{7}$$

and we have the chain

$$\tilde{H}^{1/2}(\Gamma)\subset H^{1/2}(\Gamma)\subset L^2(\Gamma)\subset\tilde{H}^{-1/2}(\Gamma)\subset H^{-1/2}(\Gamma).$$

Firstly we establish uniqueness for the direct scattering problem.

Theorem 2.1. *The problem* (3) *has at most one solution.*

Proof. Denote by B_R a sufficiently large ball with radius R containing $\bar{\Omega}$ and by ∂B_R its boundary. Let u be a solution to the homogeneous problem (3), i.e. u satisfies the problem (3) with $f=g=h=0$. It is easy to check that this solution $u\in H^1(B_R\backslash(\bar{D}\cup\bar{\Gamma}))\cup H^1(D)$ satisfies the following transmission conditions on the complementary part $\partial\Omega\backslash\bar{\Gamma}$ of $\partial\Omega$:

$$\begin{cases}u_+=u_-,\\ \dfrac{\partial u_+}{\partial\nu}=\dfrac{\partial u_-}{\partial\nu}\end{cases} \tag{8}$$

where "$\pm$" denote the limit approaching $\partial\Omega$ from outside and inside Ω respectively. Applying Green's formula for u and $\bar{u}$ in $\Omega\backslash\bar{D}$ and $B_R\backslash\bar{\Omega}$, we have

$$\int\limits_{\Omega\backslash\bar{D}}(u\Delta\bar{u}+\nabla\bar{u}\cdot\nabla u)\,dx=\int\limits_{\partial\Omega\backslash\bar{\Gamma}}u\frac{\partial\bar{u}_-}{\partial\nu}\,ds \tag{9}$$

and

$$\int_{B_R\setminus\bar\Omega} (u\Delta\bar u+\nabla\bar u\cdot\nabla u)\,dx=\int_{\Gamma} u_+\frac{\partial\bar u_+}{\partial\nu}\,ds+\int_{\partial\Omega\setminus\bar\Gamma} u_+\frac{\partial\bar u_+}{\partial\nu}\,ds+\int_{\partial B_R} u\frac{\partial\bar u}{\partial\nu}\,ds.$$

Using boundary conditions and the above transmission boundary condition (8), we have

$$\int_{\partial B_R} u\frac{\partial\bar u}{\partial\nu}\,ds=\biggl(\int_{\Omega\setminus\bar D}+\int_{B_R\setminus\bar\Omega}\biggr)\bigl(-k^2|u|^2+|\nabla u|^2\bigr)\,dx+\int_{\Gamma} ik\lambda|u_+|^2\,ds.$$

Hence

$$\operatorname{Im}\biggl(\int_{\partial B_R} u\frac{\partial\bar u}{\partial\nu}\,ds\biggr)\geqslant 0. \tag{10}$$

So, from Theorem 2.12 in [4] and a unique continuation argument we obtain that $u=0$ in $R^2\setminus(\bar D\cup\bar\Gamma)$. □

We use $[u]=u_- - u_+$ and $[\frac{\partial u}{\partial\nu}]=\frac{\partial u_-}{\partial\nu}-\frac{\partial u_+}{\partial\nu}$ to denote the jump of u and $\frac{\partial u}{\partial\nu}$ respectively, across the crack Γ. Then we have

Lemma 2.1. *If u is a solution of* (3)*, then $[u]\in\tilde H^{1/2}(\Gamma)$ and $[\frac{\partial u}{\partial\nu}]\in\tilde H^{-1/2}(\Gamma)$.*

The proof of this lemma can be found in [3].

We are now ready to prove the existence of a solution for the above scattering problem by using an integral equation approach. By Green representation formula

$$u(x)=\int_{\partial\Omega}\biggl[\frac{\partial u}{\partial\nu}\Phi(x,y)-u\frac{\partial\Phi(x,y)}{\partial\nu}\biggr]ds_y+\int_{\partial D}\biggl[\frac{\partial u}{\partial\nu}\Phi(x,y)-u\frac{\partial\Phi(x,y)}{\partial\nu}\biggr]ds_y$$

for $x\in\Omega\setminus\bar D$, and

$$u(x)=\int_{\partial\Omega}\biggl[u\frac{\partial\Phi(x,y)}{\partial\nu}-\frac{\partial u}{\partial\nu}\Phi(x,y)\biggr]ds_y$$

for $x\in R^2\setminus\bar\Omega$, where

$$\Phi(x,y)=\frac{i}{4}H_0^{(1)}\bigl(k|x-y|\bigr) \tag{11}$$

is the fundamental solution to the Helmholtz equation in R^2 and $H_0^{(1)}$ is a Hankel function of the first kind of order zero.

By making use of the known jump relations of the single- and double-layer potentials across the boundary $\partial\Omega$ (see [4] and [5]) and approaching the boundary $\partial\Omega$ from inside $\Omega\setminus\bar D$ we obtain (for $x\in\partial\Omega$)

$$u_-(x)=S_{\Omega\Omega}\frac{\partial u_-}{\partial\nu}-K_{\Omega\Omega}u_-+2\int_{\partial D}\biggl[\frac{\partial u(y)}{\partial\nu}\Phi(x,y)-u(y)\frac{\partial\Phi(x,y)}{\partial\nu}\biggr]ds_y \tag{12}$$

and

$$\frac{u_-(x)}{\partial\nu} = K'_{\Omega\Omega}\frac{\partial u_-}{\partial\nu} - T_{\Omega\Omega}u_- + 2\frac{\partial}{\partial\nu(x)}\int\limits_{\partial D}\left[\frac{\partial u(y)}{\partial\nu}\Phi(x,y) - u(y)\frac{\partial\Phi(x,y)}{\partial\nu}\right]ds_y \tag{13}$$

where $S_{\Omega\Omega}$, $K_{\Omega\Omega}$, $K^{'}_{\Omega\Omega}$ and $T_{\Omega\Omega}$ are boundary integral operators

$$S_{\Omega\Omega}: H^{-1/2}(\partial\Omega)\to H^{1/2}(\partial\Omega), \qquad K_{\Omega\Omega}: H^{1/2}(\partial\Omega)\to H^{1/2}(\partial\Omega),$$
$$K'_{\Omega\Omega}: H^{-1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega), \qquad T_{\Omega\Omega}: H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)$$

defined by

$$S_{\Omega\Omega}\varphi(x) = 2\int\limits_{\partial\Omega}\varphi(y)\Phi(x,y)\,ds_y, \qquad K_{\Omega\Omega}\varphi(x) = 2\int\limits_{\partial\Omega}\varphi(y)\frac{\Phi(x,y)}{\partial\nu_y}\,ds_y,$$
$$K'_{\Omega\Omega}\varphi(x) = 2\int\limits_{\partial\Omega}\varphi(y)\frac{\partial\Phi(x,y)}{\partial\nu_x}\,ds_y, \qquad T_{\Omega\Omega}\varphi(x) = 2\frac{\partial}{\partial\nu_x}\int\limits_{\partial\Omega}\varphi(y)\frac{\partial\Phi(x,y)}{\partial\nu_y}\,ds_y.$$

Similarly, approaching the boundary $\partial\Omega$ from inside $R^2\backslash\bar{\Omega}$ we obtain (for $x\in\partial\Omega$)

$$u_+(x) = -S_{\Omega\Omega}\frac{\partial u_+}{\partial\nu} + K_{\Omega\Omega}u_+ \tag{14}$$

and

$$\frac{u_+(x)}{\partial\nu} = -K'_{\Omega\Omega}\frac{\partial u_+}{\partial\nu} + T_{\Omega\Omega}u_+. \tag{15}$$

From (12)–(15), restricting u to boundary Γ^- we have

$$\begin{aligned}2g(x) = {} & S_{\Omega\Omega}\left(\frac{\partial u_-}{\partial\nu} - \frac{\partial u_+}{\partial\nu}\right)\bigg|_{\Gamma} - K_{\Omega\Omega}(u_- - u_+)|_{\Gamma} + (u_- - u_+)|_{\Gamma}\\ & + 2\int\limits_{\partial D}\frac{\partial u}{\partial\nu}\Phi\,ds|_{\Gamma^-} - 2\int\limits_{\partial D}\frac{\partial\Phi}{\partial\nu}u\,ds|_{\Gamma^-}, \quad x\in\Gamma^-,\end{aligned} \tag{16}$$

where $(\cdot)|_{\Gamma^-}$ means a restriction to Γ^-.

Define

$$S_{D\Gamma}\varphi(x) = 2\int\limits_{\partial D}\varphi(y)\Phi(x,y)\,ds_y|_{\Gamma^-}, \tag{17}$$

$$S_{\Omega\Gamma}\varphi(x) = 2\int\limits_{\partial\Omega}\varphi(y)\Phi(x,y)\,ds_y|_{\Gamma^-}, \tag{18}$$

$$K_{\Omega\Gamma}\varphi(x) = 2\int\limits_{\partial\Omega}\frac{\partial\Phi(x,y)}{\partial\nu}\varphi(y)\,ds_y|_{\Gamma^-}, \tag{19}$$

and

$$\left.\frac{\partial u}{\partial \nu}\right|_{\partial D} = a, \qquad \left.\left(\frac{\partial u_-}{\partial \nu} - \frac{\partial u_+}{\partial \nu}\right)\right|_{\Gamma} = b, \qquad (u_- - u_+)|_{\Gamma} = c. \tag{20}$$

Then zero extends b and c to the whole $\partial\Omega$ in the following

$$\tilde{b} = \begin{cases} 0 & \text{on } \partial\Omega\backslash\Gamma, \\ b & \text{on } \Gamma, \end{cases} \qquad \tilde{c} = \begin{cases} 0 & \text{on } \partial\Omega\backslash\Gamma, \\ c & \text{on } \Gamma. \end{cases}$$

By using the boundary conditions in (3), we rewrite (16) as

$$S_{D\Gamma}a + S_{\Omega\Gamma}\tilde{b} - K_{\Omega\Gamma}\tilde{c} + c = r_1(x) \tag{21}$$

where

$$r_1(x) = 2g(x) + 2\int\limits_{\partial D} \frac{\partial \Phi}{\partial \nu} f(y)\, ds|_{\Gamma^-}, \quad x \in \Gamma^-. \tag{22}$$

For simplicity, we modify (21) as

$$S_{D\Gamma}a + S_{\Gamma\Gamma}b - K_{\Gamma\Gamma}c + c = r_1(x) \tag{23}$$

where the operator $S_{\Gamma\Gamma}$ is the operator applied to a function with supp $\subseteq \bar{\Gamma}$ and evaluated on Γ, with analogous definition for $K_{\Gamma\Gamma}$. We have mapping properties (see [2] and [3])

$$S_{D\Gamma} : \tilde{H}^{-1/2}(\partial D) \to H^{1/2}(\Gamma), \tag{24}$$

$$S_{\Gamma\Gamma} : \tilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma), \tag{25}$$

$$K_{\Gamma\Gamma} : \tilde{H}^{1/2}(\Gamma) \to H^{1/2}(\Gamma). \tag{26}$$

From (12) and (13), we have

$$\begin{aligned} -ik\lambda\left(S_{\Omega\Omega}\frac{\partial u_-}{\partial \nu} - K_{\Omega\Omega}u_-\right) &= -ik\lambda\left[u_- - 2\int\limits_{\partial D}\left(\frac{\partial u}{\partial \nu}\Phi - u\frac{\partial \Phi}{\partial \nu}\right) ds\right] \\ &= -ik\lambda u_- + 2ik\lambda\int\limits_{\partial D}\frac{\partial u}{\partial \nu}\Phi\, ds - 2ik\lambda\int\limits_{\partial D} f\frac{\partial \Phi}{\partial \nu}\, ds, \end{aligned} \tag{27}$$

$$-K'_{\Omega\Omega}\frac{\partial u_-}{\partial \nu} + T_{\Omega\Omega}u_- = -\frac{\partial u_-}{\partial \nu} + 2\frac{\partial}{\partial \nu(x)}\int\limits_{\partial D}\frac{\partial u}{\partial \nu}\Phi\, ds - 2\frac{\partial}{\partial \nu(x)}\int\limits_{\partial D} f\frac{\partial \Phi}{\partial \nu}\, ds \tag{28}$$

and

$$-ik\lambda u_- - \frac{\partial u_-}{\partial \nu} = -ik\lambda(u_- - u_+) - \left(\frac{\partial u_-}{\partial \nu} - \frac{\partial u_+}{\partial \nu}\right) - ik\lambda u_+ - \frac{\partial u_+}{\partial \nu}. \tag{29}$$

Using (14) and (15), the above formulas (27) and (28) imply that

$$
\begin{aligned}
\frac{\partial u_+}{\partial\nu}+ik\lambda u_+ &= -K'_{\Omega\Omega}\frac{\partial u_+}{\partial\nu}+T_{\Omega\Omega}u_+ + ik\lambda\left(K_{\Omega\Omega}u_+ - S_{\Omega\Omega}\frac{\partial u_+}{\partial\nu}\right)\\
&= K'_{\Omega\Omega}\left(\frac{\partial u_-}{\partial\nu}-\frac{\partial u_+}{\partial\nu}\right)-T_{\Omega\Omega}(u_- - u_+)+ik\lambda S_{\Omega\Omega}\left(\frac{\partial u_-}{\partial\nu}-\frac{\partial u_+}{\partial\nu}\right)\\
&\quad - ik\lambda K_{\Omega\Omega}(u_- - u_+) - ik\lambda\left(S_{\Omega\Omega}\frac{\partial u_-}{\partial\nu}-K_{\Omega\Omega}u_-\right)-K'_{\Omega\Omega}\frac{\partial u_-}{\partial\nu}+T_{\Omega\Omega}u_-\\
&= K'_{\Omega\Omega}\left(\frac{\partial u_-}{\partial\nu}-\frac{\partial u_+}{\partial\nu}\right)-T_{\Omega\Omega}(u_- - u_+)+ik\lambda S_{\Omega}\left(\frac{\partial u_-}{\partial\nu}-\frac{\partial u_+}{\partial\nu}\right)\\
&\quad - ik\lambda K_{\Omega\Omega}(u_- - u_+) - ik\lambda u_- - \frac{\partial u_-}{\partial\nu} + 2ik\lambda\int_{\partial D}\frac{\partial u}{\partial\nu}\Phi\,ds\\
&\quad + 2\frac{\partial}{\partial\nu(x)}\int_{\partial D}\frac{\partial u}{\partial\nu}\Phi\,ds - 2ik\lambda\int_{\partial D} f\frac{\partial\Phi}{\partial\nu}\,ds - 2\frac{\partial}{\partial\nu(x)}\int_{\partial D} f\frac{\partial\Phi}{\partial\nu}\,ds.
\end{aligned}
$$

Then using (29) we have

$$
\begin{aligned}
2\left(\frac{\partial u_+}{\partial\nu}+ik\lambda u_+\right) &= K'_{\Omega\Omega}\left(\frac{\partial u_-}{\partial\nu}-\frac{\partial u_+}{\partial\nu}\right)-T_{\Omega\Omega}(u_- - u_+)+ik\lambda S_{\Omega\Omega}\left(\frac{\partial u_-}{\partial\nu}-\frac{\partial u_+}{\partial\nu}\right)\\
&\quad - ik\lambda K_{\Omega\Omega}(u_- - u_+) - 2ik\lambda(u_- - u_+) - \left(\frac{\partial u_-}{\partial\nu}-\frac{\partial u_+}{\partial\nu}\right)\\
&\quad + 2ik\lambda\int_{\partial D}\frac{\partial u}{\partial\nu}\Phi\,ds + 2\frac{\partial}{\partial\nu(x)}\int_{\partial D}\frac{\partial u}{\partial\nu}\Phi\,ds\\
&\quad - 2ik\lambda\int_{\partial D} f\frac{\partial\Phi}{\partial\nu}\,ds - 2\frac{\partial}{\partial\nu(x)}\int_{\partial D} f\frac{\partial\Phi}{\partial\nu}\,ds.
\end{aligned}
\tag{30}
$$

Restricting (30) to Γ^+ and using (16) we have

$$
\begin{aligned}
&2\frac{\partial}{\partial\nu(x)}\int_{\partial D}\frac{\partial u}{\partial\nu}\Phi\,ds|_{\Gamma^+} + K'_{\Omega\Omega}\left(\frac{\partial u_-}{\partial\nu}-\frac{\partial u_+}{\partial\nu}\right)\bigg|_{\Gamma} - T_{\Omega\Omega}(u_- - u_+)|_{\Gamma}\\
&\quad - \left(\frac{\partial u_-}{\partial\nu}-\frac{\partial u_+}{\partial\nu}\right)\bigg|_{\Gamma} - 2ik\lambda(u_- - u_+)|_{\Gamma} = r_2(x)
\end{aligned}
\tag{31}
$$

where

$$
r_2(x) = 2h(x) - ik\lambda r_1(x) + 2ik\lambda\int_{\partial D} f\frac{\partial\Phi}{\partial\nu}\,ds + 2\frac{\partial}{\partial\nu(x)}\int_{\partial D} f\frac{\partial\Phi}{\partial\nu}\,ds
$$

for $x\in\Gamma^+$.

Define

$$
K'_{D\Gamma}\varphi(x) = 2\frac{\partial}{\partial\nu(x)}\int_{\partial D}\varphi(x)\Phi\,ds|_{\Gamma^+},
\tag{32}
$$

and use the notation in (20), we can rewrite (31) as

$$K'_{D\Gamma}a + K'_{\Gamma\Gamma}b - T_{\Gamma\Gamma}c - b - 2ik\lambda c = r_2(x) \tag{33}$$

where the operators $K'_{\Gamma\Gamma}$ and $T_{\Gamma\Gamma}$ are restriction operators (see (23)). As before, we have mapping properties

$$K'_{\Gamma\Gamma} : \tilde{H}^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma), \tag{34}$$

$$T_{\Gamma\Gamma} : \tilde{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma). \tag{35}$$

By using Green formula and approaching the boundary ∂D from inside $\Omega\backslash\bar{D}$ we obtain (for $x \in \partial D$)

$$\begin{aligned} u(x) = {} & 2\int_{\partial D} \frac{\partial u(y)}{\partial \nu}\Phi(x,y)\,ds_y + 2\int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu}u(y)\,ds_y \\ & + 2\int_{\partial\Omega} \left[\frac{\partial u_-(y)}{\partial \nu}\Phi(x,y) - u_-(y)\frac{\partial \Phi(x,y)}{\partial \nu}\right] ds_y. \end{aligned} \tag{36}$$

The last term in (36) can be reformulated as

$$\begin{aligned} & 2\int_{\partial\Omega} \left[\frac{\partial u_-(y)}{\partial \nu}\Phi(x,y) - u_-(y)\frac{\partial \Phi(x,y)}{\partial \nu}\right] ds_y \\ & = 2\int_{\partial\Omega} \left[\left(\frac{\partial u_-(y)}{\partial \nu} - \frac{\partial u_+(y)}{\partial \nu}\right)\Phi(x,y) - \big(u_-(y) - u_+(y)\big)\frac{\partial \Phi(x,y)}{\partial \nu}\right] ds_y \\ & \quad + 2\int_{\partial\Omega} \left[\frac{\partial u_+(y)}{\partial \nu}\Phi(x,y) - u_+(y)\frac{\partial \Phi(x,y)}{\partial \nu}\right] ds_y. \end{aligned} \tag{37}$$

Since $x \in \partial D$ and $y \in \partial\Omega$ in (37), we have the following result (see [4]).

Lemma 2.2. *By using Green formula and the Sommerfeld radiation condition* (2), *we obtain*

$$\int_{\partial\Omega} \left[\frac{\partial u_+(y)}{\partial \nu}\Phi(x,y) - u_+(y)\frac{\partial \Phi(x,y)}{\partial \nu}\right] ds_y = 0. \tag{38}$$

Proof. Denote by B_R a sufficiently large ball with radius R containing $\bar{\Omega}$ and use Green formula inside $B_R\backslash\bar{\Omega}$. Furthermore notice that $x \in \partial D$, $y \in \partial\Omega$ and using the Sommerfeld radiation condition (2), we can prove this lemma. □

From (36), (37) and the above lemma, we have

$$\begin{aligned} & 2\int_{\partial D} \frac{\partial u(y)}{\partial \nu}\Phi(x,y)\,ds_y|_{\partial D} + 2\int_{\partial\Omega} \left[\frac{\partial u_-(y)}{\partial \nu} - \frac{\partial u_+(y)}{\partial \nu}\right]\Phi(x,y)\,ds_y|_{\partial D} \\ & \quad - 2\int_{\partial\Omega} \big[u_-(y) - u_+(y)\big]\frac{\partial \Phi(x,y)}{\partial \nu}\,ds_y|_{\partial D} = r_0(x) \end{aligned} \tag{39}$$

where

$$r_0(x) = f(x) - 2\int_{\partial D} f(y)\frac{\partial \Phi(x,y)}{\partial \nu}\,ds_y|_{\partial D}. \tag{40}$$

Define

$$S_{DD}\varphi(x) = 2\int\limits_{\partial D} \varphi(y)\Phi(x, y)\, ds_y|_{\partial D}, \tag{41}$$

and use the notation in (20), we can rewrite (39) as

$$S_{DD}a + S_{\Gamma D}b - K_{\Gamma D}c = r_0(x). \tag{42}$$

Similarly, $S_{\Gamma D}$ and $K_{\Gamma D}$ are restriction operators as before, and we have mapping properties

$$S_{DD} : \tilde{H}^{-1/2}(\partial D) \to H^{1/2}(\partial D), \tag{43}$$

$$S_{\Gamma D} : \tilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\partial D), \tag{44}$$

$$K_{\Gamma D} : \tilde{H}^{1/2}(\Gamma) \to H^{1/2}(\partial D). \tag{45}$$

If we define

$$A = \begin{pmatrix} S_{DD} & S_{\Gamma D} & -K_{\Gamma D} \\ S_{D\Gamma} & S_{\Gamma\Gamma} & I - K_{\Gamma\Gamma} \\ K'_{D\Gamma} & K'_{\Gamma\Gamma} - I & -(T_{\Gamma\Gamma} + 2ik\lambda I) \end{pmatrix} \tag{46}$$

and

$$\vec{p} = \begin{pmatrix} r_0(x) \\ r_1(x) \\ r_2(x) \end{pmatrix}, \tag{47}$$

then combine (23), (33) and (42), we have a boundary integral system

$$A\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \vec{p}. \tag{48}$$

Remark. If the above system has a unique solution, our problem (3) will have a unique solution (see [4] and [5]).

3. Existence and uniqueness

Based on the idea of the paper [3], we show the existence and uniqueness of a solution to the integral system (48).

Theorem 3.1. *The boundary integral system* (48) *has a unique solution.*

Proof. There are two main steps to prove the theorem: Step 1 is to show that the operator A is Fredholm with index zero; Step 2 is to prove that $\operatorname{Kern} A = \{0\}$.

Step 1. As we know, the operator S_{DD} is positive and bounded below up to a compact perturbation (see [17]), that is, there exists a compact operator

$$L_D : H^{-1/2}(\partial D) \to H^{1/2}(\partial D) \tag{49}$$

such that

$$\operatorname{Re}\big(\big\langle (S_{DD}+L_D)\psi,\bar{\psi}\big\rangle\big) \geqslant C\|\psi\|^2_{H^{-1/2}(\partial D)}, \quad \text{for } \psi \in H^{-1/2}(\partial D) \tag{50}$$

where $\langle,\rangle$ denotes the duality between $H^{-1/2}(\partial D)$ and $H^{1/2}(\partial D)$.

For convenience, in the following discussion we define

$$S_0 = S_{DD} + L_D. \tag{51}$$

Similarly, the operators $S_{\Omega\Omega}$ and $-T_{\Omega\Omega}$ are positive and bounded below up to compact perturbations (see [17]), that is, there exist compact operators

$$L_\Omega : H^{-1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega), \tag{52}$$

$$L_T : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega) \tag{53}$$

such that

$$\operatorname{Re}\big(\big\langle (S_{\Omega\Omega}+L_\Omega)\psi,\bar{\psi}\big\rangle\big) \geqslant C\|\psi\|^2_{H^{-1/2}(\partial D)}, \quad \text{for } \psi \in H^{-1/2}(\partial\Omega), \tag{54}$$

$$\operatorname{Re}\big(\big\langle -(T_{\Omega\Omega}+L_T)\varphi,\bar{\varphi}\big\rangle\big) \geqslant C\|\varphi\|^2_{H^{1/2}(\partial D)}, \quad \text{for } \varphi \in H^{1/2}(\partial\Omega). \tag{55}$$

Define $S_1 = S_{\Omega\Omega} + L_\Omega$ and $T_1 = -(T_{\Omega\Omega} + L_T)$, then S_1 and T_1 are bounded below up and positive.

Take $a \in H^{-1/2}(\partial D)$ and let $\tilde{b} \in H^{-1/2}(\partial\Omega)$ and $\tilde{c} \in H^{1/2}(\partial\Omega)$ be the extension by zero to $\partial\Omega$ of $b \in \tilde{H}^{-1/2}(\Gamma)$ and $c \in \tilde{H}^{1/2}(\Gamma)$ respectively.

Denote $\xi = (a, b, c)^T$,

$$H = H^{-1/2}(\partial D) \times \tilde{H}^{-1/2}(\Gamma) \times \tilde{H}^{1/2}(\Gamma), \tag{56}$$

$$H^* = H^{1/2}(\partial D) \times \tilde{H}^{1/2}(\Gamma) \times \tilde{H}^{-1/2}(\Gamma). \tag{57}$$

It is easy to check that the operators $S_{\Gamma D}$, $S_{D\Gamma}$, $K_{\Gamma D}$ and $K'_{D\Gamma}$ are compact operators, then we can rewrite $A\xi$ as the following:

$$A\xi = A\begin{pmatrix} a \\ b \\ c \end{pmatrix} = A_0\begin{pmatrix} a \\ b \\ c \end{pmatrix} + A_C\begin{pmatrix} a \\ b \\ c \end{pmatrix} = A_0\xi + A_C\xi \tag{58}$$

with

$$A_C\xi = \begin{pmatrix} -L_D a + S_{\Gamma D} b - K_{\Gamma D} c \\ S_{D\Gamma} a - L_\Omega b \\ K'_{D\Gamma} a + L_T c \end{pmatrix} \tag{59}$$

and

$$A_0\xi = \begin{pmatrix} S_0 a \\ S_1 b - (K_{\Gamma\Gamma} - I)c \\ (K'_{\Gamma\Gamma} - I)b + T_1 c - 2ik\lambda c \end{pmatrix} \tag{60}$$

where $A_C : H \to H^*$ is compact and $A_0 : H \to H^*$ defines a sesquilinear form, i.e.

$$\langle A_0\xi, \bar{\xi}\rangle_{H,H^*} = (S_0 a, a) + (S_1\tilde{b}, \tilde{b}) - (K_{\Gamma\Gamma}c, b)|_\Gamma + (c, b) \\ + \left(K'_{\Gamma\Gamma}b, c\right)\big|_\Gamma - (b, c) + (T_1\tilde{c}, \tilde{c}) - 2ik\lambda(c, c). \tag{61}$$

Here (u, v) denotes the scalar product on $L^2(\partial D)$ or $L^2(\partial\Omega)$ defined by $\int_{\partial D} u\bar{v}\,ds$ or $\int_{\partial\Omega} u\bar{v}\,ds$, and $(u, v)|_\Gamma$ is the scalar product on $L^2(\Gamma)$.

By properties of the operators S_0, S_1 and T_1, we have

$$\mathrm{Re}(S_0 a, a) \geqslant c_1\|a\|^2_{H^{-1/2}(\partial D)}, \tag{62}$$

$$\mathrm{Re}(S_1\tilde{b}, \tilde{b}) \geqslant c_2\|b\|^2_{\tilde{H}^{-1/2}(\Gamma)} \tag{63}$$

and

$$\mathrm{Re}(T_1\tilde{c}, \tilde{c}) \geqslant c_3\|c\|^2_{\tilde{H}^{1/2}(\Gamma)}. \tag{64}$$

Similarly,

$$\mathrm{Re}\left[\left(K'_{\Gamma\Gamma}b, c\right)\big|_\Gamma - (K_{\Gamma\Gamma}c, b)|_\Gamma\right] = \mathrm{Re}\left[(b, K_{\Gamma\Gamma}c)|_\Gamma - (K_{\Gamma\Gamma}c, b)|_\Gamma\right] \\ = \mathrm{Re}\left[\overline{(K_{\Gamma\Gamma}c, b)}|_\Gamma - (K_{\Gamma\Gamma}c, b)|_\Gamma\right] = 0 \tag{65}$$

and

$$\mathrm{Re}\left[(c, b) - (b, c) - 2ik\lambda(c, c)\right] = \mathrm{Re}\left[(c, b) - \overline{(c, b)} - 2ik\lambda(c, c)\right] = 0. \tag{66}$$

So the operator A_0 is coercive, i.e.

$$\mathrm{Re}\left(\left\langle (A - A_C)\xi, \bar{\xi}\right\rangle_{H,H^*}\right) \geqslant C\|\xi\|^2_H, \quad \text{for } \xi \in H, \tag{67}$$

whence the operator A is Fredholm with index zero.

Step 2: In this part, we show that $\mathrm{Kern}\, A = \{0\}$. To this end let $\psi = (a, b, c)^T \in H$ be a solution of the homogeneous equation $A\psi = 0$, we want to prove that $\psi \equiv 0$.

However, $A\psi = 0$ means that

$$\begin{cases} S_{DD}a + S_{\Gamma D}b - K_{\Gamma D}c = 0, \\ S_{D\Gamma}a + S_{\Gamma\Gamma}b + (I - K_{\Gamma\Gamma})c = 0, \\ K'_{D\Gamma}a + \left(K'_{\Gamma\Gamma} - I\right)b - (T_{\Gamma\Gamma} + 2ik\lambda I)c = 0. \end{cases} \tag{68}$$

Define the potential

$$v(x) = S_D a + S_\Omega\tilde{b} - K_\Omega\tilde{c} \tag{69}$$

where $\tilde{b}$ and $\tilde{c}$ have the same meaning as before and

$$S_D\varphi(x) = \int_{\partial D} \varphi(y)\Phi(x, y)\,ds_y, \tag{70}$$

$$S_\Omega\varphi(x) = \int_{\partial\Omega} \varphi(y)\Phi(x, y)\,ds_y, \tag{71}$$

$$K_\Omega\varphi(x) = \int_{\partial\Omega} \varphi(y)\frac{\partial\Phi(x, y)}{\partial\nu(y)}\,ds_y. \tag{72}$$

This potential $v(x)$ satisfies Helmholtz equation in $R^2\backslash(\partial D\cup\bar{\Gamma})$ and the Sommerfeld radiation condition.

Consider the potential $v(x)$ inside $\Omega\backslash\bar{D}$ and approach the boundary ∂D ($x\to\partial D$), we have

$$v(x)=\frac{1}{2}\{S_{DD}a+S_{\Gamma D}b-K_{\Gamma D}c\},\quad x\in\partial D. \tag{73}$$

The first formula in (68) implies that

$$v(x)|_{\partial D}=0. \tag{74}$$

Similarly, consider the potential $v(x)$ inside $\Omega\backslash\bar{D}$ and approach the boundary $\partial\Omega$ ($x\to\partial\Omega$), then restrict $v(x)$ to the partial boundary Γ, we have

$$v(x)|_{\Gamma}=\frac{1}{2}\{S_{D\Gamma}a+S_{\Gamma\Gamma}b-K_{\Gamma\Gamma}c+c\},\quad x\in\Gamma. \tag{75}$$

The second formula in (68) implies that

$$v(x)|_{\Gamma^-}=0. \tag{76}$$

Now, we consider the potential $v(x)$ in the region $R^2\backslash\bar{\Omega}$ and approach the boundary $\partial\Omega$ ($x\to\partial\Omega$), then restrict $v(x)$ to the partial boundary Γ, we have

$$v(x)|_{\Gamma}=\frac{1}{2}\{S_{D\Gamma}a+S_{\Gamma\Gamma}b-K_{\Gamma\Gamma}c-c\},\quad x\in\Gamma, \tag{77}$$

and

$$\left.\frac{\partial v(x)}{\partial\nu}\right|_{\Gamma}=\frac{1}{2}\left\{K'_{D\Gamma}a+K'_{\Gamma\Gamma}b-b-T_{\Gamma\Gamma}c\right\},\quad x\in\Gamma. \tag{78}$$

Combine (77) and (78), then use the second and third formula in (68), on Γ^+ we have

$$\begin{aligned}\left.\left(\frac{\partial v(x)}{\partial\nu}+ik\lambda v(x)\right)\right|_{\Gamma^+}&=\frac{1}{2}\left\{\left[K'_{D\Gamma}a+K'_{\Gamma\Gamma}b-b-T_{\Gamma\Gamma}c\right]+ik\lambda[S_{D\Gamma}a+S_{\Gamma\Gamma}b-K_{\Gamma\Gamma}c-c]\right\}\\&=\frac{1}{2}\left\{2ik\lambda c+ik\lambda[S_{D\Gamma}a+S_{\Gamma\Gamma}b-K_{\Gamma\Gamma}c-c]\right\}\\&=\frac{1}{2}\left\{ik\lambda[S_{D\Gamma}a+S_{\Gamma\Gamma}b-K_{\Gamma\Gamma}c+c]\right\}\\&=0.\end{aligned} \tag{79}$$

From (74), (76) and (79), the potential $v(x)$ satisfies the following boundary value problem

$$\begin{cases}\Delta v+k^2v=0 & \text{in } R^2\backslash(\bar{D}\cup\Gamma),\\ v_-=0 & \text{on } \Gamma^-,\\ \dfrac{\partial v_+}{\partial\nu}+ik\lambda v_+=0 & \text{on } \Gamma^+,\\ v=0 & \text{on } \partial D,\end{cases} \tag{80}$$

and the Sommerfeld radiation condition

$$\lim_{\nu\to\infty}\sqrt{r}\left(\frac{\partial v}{\partial r}-ikv\right)=0 \tag{81}$$

uniformly in $\hat{x}=x/|x|$ with $r=|x|$.

The uniqueness result Theorem 2.1 in Section 2 implies that

$$v(x)=0, \quad x\in R^2\backslash(D\cup\Gamma). \tag{82}$$

Notice that $-k^2$ is not Dirichlet eigenvalue of the Laplace operator in D, so

$$v(x)=0, \quad x\in D. \tag{83}$$

Therefore, the well-known jump relations imply that

$$a=\left(\frac{\partial v_-}{\partial \nu}-\frac{\partial v_+}{\partial \nu}\right)\bigg|_{\partial D}=0 \tag{84}$$

where $\frac{\partial v_\pm}{\partial \nu}$ mean the limit approach to ∂D from outside and inside respectively.

Furthermore

$$b=\left(\frac{\partial v_-}{\partial \nu}-\frac{\partial v_+}{\partial \nu}\right)\bigg|_{\Gamma}=0 \tag{85}$$

and

$$c=(v_- - v_+)|_\Gamma=0. \tag{86}$$

So

$$\psi=(a,b,c)^T=(0,0,0)^T. \tag{87}$$

In conclusion, we complete the proof of the theorem. □

Remark. If we remove the condition "$-k^2$ is not Dirichlet eigenvalue of the Laplace operator in D", instead of it by the assumption that $\mathrm{Im}\,k>0$, Theorem 2.1 in Section 2 and Theorem 3.1 in Section 3 are also true.

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